STEADY-STATE FREEZING OF A LIQUID DURING LAMINAR FLOW THROUGH PIPES

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The temperature fields of the liquid phase and of the solid phase, and the liquid—solid inter-phase boundary, are determined under conditions of forced flow through circular pipes.

Among nonlinear problems arising from the energy equation and from the equation of heat conduction there stand out the problems of the Stefan kind with their many practical applications. A survey of studies on this kind of problems has been made in [1]. The authors of [2] have succeeded in demonstrating that, with the assumption of a parabolic velocity profile, the problem of steady-state freezing in a circular pipe reduces to the Graetz problem.

We will consider the steady-state freezing of a liquid which flows through a pipe of circular cross section.

We assume that the pipe is filled with liquid and that the temperature of the liquid as well as the temperature of the pipe wall are at the freezing point t_f. A laminar stream of liquid comes in through the entrance section, where the origin of coordinates will be located, at a constant velocity v_0 and a constant temperature t_0. As the thermal wave front moves through the liquid, let the temperature of the pipe wall drop to a constant level t_c below the freezing point t_f.

The problem will now be formulated with the following assumptions:

a) heat conduction along the pipe axis through the liquid and the solid is negligible;

b) the thermophysical properties of both phases are constant;

c) the flow through the pipe is forced, the axial component of velocity being given by the relation $x_X = v_0 r_0^2 / \xi^2(x)$.

With these assumptions, then, the energy equation for the liquid phase is

$$\frac{1}{\delta^2} \cdot \frac{\partial T_1}{\partial X} = \frac{\partial^2 T_1}{\partial R^2} + \frac{1}{R} \cdot \frac{\partial T_1}{\partial R} \qquad (1)$$

in dimensionless form and the boundary conditions are

$$T_1(R,\ 0) = 1; \quad T_1(\delta,\ X) = 0; \quad \left(\frac{\partial T_1}{\partial R} \right)_{R=0} = 0. \qquad (2)$$

The equation of heat conduction for the solid phase can be written as

$$\frac{d}{dR} \left[R \frac{dT_2}{dR} \right] = 0 \qquad (3)$$

and the boundary conditions as

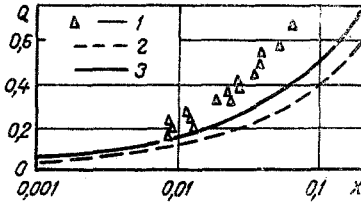$$T_2(1,\ X) = 1; \quad T_2(\delta,\ X) = 0. \qquad (4)$$
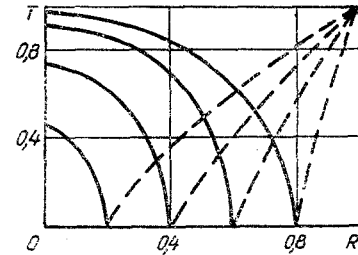
Fig. 1   Fig. 2   Fig. 3

Fig. 1. Profiles of interphase boundary as a function of the coordinate X.

Fig. 2. Comparison between test values and theoretical values of thermal fluxes: test values according to [2] (1); calculated values according to [2] (2); calculated values according to formula (11) (3).

Fig. 3. Temperature distribution in the liquid (solid lines) and in the solid layer (dashed lines).

The following condition is satisfied on the liquid—solid interface:

$$\frac{\partial T_1}{\partial R}\bigg|_{R=\delta} + B\,\frac{\partial T_2}{\partial R}\bigg|_{R=\delta} = 0. \tag{5}$$

It is assumed here, moreover, that $\delta(x)$ becomes neither zero nor unity. Equation (1) with the boundary conditions (2) will be solved by the Grinberg method [3], which is based on a series expansion of the solution in terms of "local" eigenfunctions of the corresponding boundary-value problem. The temperature field of the liquid phase will be expressed in terms of a Fourier—Bessel series

$$T_1 = \frac{2}{\delta^2} \sum_{n=1}^{\infty} U_n\,\frac{J_0(R\gamma_n/\delta)}{J_1(\gamma_n)}\,, \tag{6}$$

where the coefficients are defined as follows:

$$U_n(X) = \int_0^\delta T_1 R J_0(R\gamma_n/\delta)\,dR$$

and γ_n are the roots of the characteristic equation $J_0(\gamma) = 0$.

The coefficients of the series are determined by the system of ordinary differential equations:

$$\frac{dU_n}{dX} + \gamma_n^2 U_n = \frac{1}{\delta}\cdot\frac{d\delta}{dX}\sum_{m=1}^{\infty}\alpha_{nm}U_m, \tag{7}$$

where

$$U_n(0) = \frac{\delta^2(0)}{\gamma_n}\,J_1(\gamma_n),$$

$$\alpha_{nm} = \frac{2\gamma_n J_1(\gamma_n)\,\gamma_m}{J_1(\gamma_m)\,(\gamma_m^2 - \gamma_n^2)} \quad \text{for} \quad m \pm n, \tag{8}$$

$$\alpha_{mn} = 1 \quad \text{for} \quad m = n.$$

The solution to Eq. (3) with the boundary conditions (4) is

$$T_2 = 1 - \frac{\ln R}{\ln \delta}\,. \tag{9}$$

The partial derivatives in (5) can be determined from (6) and (9), respectively, by differentiating with respect to R and evaluating for $R = \delta$. The equation of the interphase boundary profile will then be

$$\frac{\delta^2}{\ln \delta} = -\frac{2}{B} \sum_{n=1}^{\infty} \frac{U_n \gamma_n}{J_1(\gamma_n)} \ . \tag{10}$$

The expression for the thermal flux in the liquid approaching a pipe segment of length X is

$$Q = 2 \int_0^X \left[-\delta \frac{\partial T_1}{\partial X} (\delta, X) \right] dX. \tag{11}$$

The system of ordinary differential equations (7), (8) and Eq. (10) is nonlinear. It was solved numerically on a MINSK-22 computer by the method of finite differences. Series (6) and (10) converge fast. The computations have shown that four terms of the series suffice. The values for $B = 1.786 \gamma_n$ and $J_1(\gamma_n)$ were taken from [4].

The liquid—solid interface is shown graphically in Fig. 1, as a function of the dimensionless coordinate X. The calculated steady-state thermal flux in the liquid is compared in Fig. 2 with experimental and theoretical results from [2]. At a given mode of forced flow, the magnitude of the thermal flux is somewhere between the theoretical and the experimental values in [2]. The temperature distributions in the liquid stream and in the solid layer are shown graphically in Fig. 3.

NOTATION

t	is the temperature;
v	is the velocity of liquid;
x	is the axial coordinate;
r	is the radial coordinate;
ξ	is the radius of liquid—solid interface;
a	is the thermal diffusivity of liquid;
λ	is the thermal conductivity;
$T_1 = (t_1 - t_f)/(t_0 - t_f)$	is the dimensionless temperature of liquid phase;
$T_2 = (t_2 - t_f)/(t_c - t_f)$	is the dimensionless temperature of solid phase;
$Pe = 2v_0 r_0 / a$	is the Peclet number;
$X = 2x/r_0 Pe$	is the dimensionless axial coordinate;
$R = r/r_0$	is the dimensionless radial coordinate;
$\delta = \xi/r_0$	is the dimensionless radius of interface;
$B = \lambda_2 (t_f - t_c)/\lambda_1 (t_0 - f_t)$	is the dimensionless parameter of freezing;
q	is the thermal flux;
c	is the specific heat of liquid;
ρ	is the density of liquid;
$Q = q/\pi v_0 r_0^2 c\rho(t_0 - t_f)$	is the dimensionless thermal flux.

LITERATURE CITED

1. L. I. Rubinshtein, The Stefan Problem [in Russian], Zvaigzne, Riga (1967).
2. P. Zerkley and I. Sunderland, Heat Transfer, No. 2 (1961).
3. G. A. Grinberg, Prikl. Matem. i Mekh., 31, No. 2 (1967).
4. E. Jahnke, F. Emde, and F. Lesh, Special Functions [Russian translation], Nauka, Moscow (1968).